Dual families of non-commutative quantum systems

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We demonstrate how a one parameter family of interacting non-commuting Hamiltonians, which are physically equivalent, can be constructed in non-commutative quantum mechanics. This construction is carried out exactly (to all orders in the non-commutative parameter) and analytically in two dimensions for a free particle and a harmonic oscillator moving in a constant magnetic field. We discuss the significance of the Seiberg-Witten map in this context. It is shown for the harmonic oscillator potential that an approximate duality, valid in the low energy sector, can be constructed between the interacting commutative and a non-interacting non-commutative Hamiltonian. This approximation holds to order $1/B$ and is therefore valid in the case of strong magnetic fields and weak Landau-level mixing.

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I. INTRODUCTION

Non-commutative quantum field theories [1] have been studied extensively because of its direct connection to certain low energy limits of string theory [2]. Non-commutative spaces can arise as brane configurations in string theory and in the matrix model of M-theory [4]. Despite the large body of literature on non-commutative quantum field theories, relatively few works explore the quantum mechanics of particles on non-commutative spaces [5, 6, 7].

Recently the Seiberg-Witten map has begun to play a central role in the analysis of non-commutative quantum field theories. The rational behind this map derives from the observation that commutative and non-commutative field theories result from different regularizations of the same gauge theory, at least in two dimensions. Thus, there should exist a map between these theories which reflects the fact that the physical content of the two theories is the same. The Seiberg-Witten map is therefore a map from the non-commutative to the commutative space which preserves the gauge invariance and the physics [2]. However, due to its classical nature, it is not clear whether this map will hold at the quantum level or not [8]. It is therefore natural to enquire about the status of this map in non-commutative quantum mechanics where, apart from a few works [6, 9] which consider the Seiberg-Witten map only to lowest order in the non-commutative parameter, very little has been done.

A second motivation for the present work comes from the by now well known non-commutative paradigm associated with the quantum Hall effect [10, 11]. In particular [12] explores the possibility that the quantum Hall effect in non-commutative space can serve as a paradigm for the fractional quantum Hall effect. On the other hand it was realized immediately after the discovery of the fractional quantum Hall effect that the Coulomb interaction plays an essential role in the understanding of this phenomenon [13]. This raises the question whether the non-commutative Hamiltonian introduced by [12] in a somewhat ad hoc way can be reinterpreted as an effective non-commutative Hamiltonian which describes the same physics as the interacting commutative theory, at least in some approximation. Clearly, this equivalence cannot be exact as it is well known [5, 6] that a non-interacting commutative Hamiltonian with constant magnetic field maps onto a non-interacting non-commutative Hamiltonian with constant magnetic field. However, one might contemplate the possibility that there is some preferred value of the non-commutative parameter which minimizes the interaction on the non-commutative level. If this is the case the corresponding non-interacting non-commutative Hamiltonian might be a good starting point for a computation which treats the residual interaction as a perturbation. This might seem problematic due to the degeneracy of the Landau levels. However, under the assumption of a central potential this construction can be carried out in each angular momentum sector, which effectively lifts this degeneracy and allows for a perturbative treatment in each sector (see section VI).

Keeping the philosophy of the above remarks in

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mind, i.e. the physical equivalence of different non-commutative descriptions, the following question poses itself quite naturally: how should a family of non-commutative Hamiltonians be parameterized as a function of the non-commutative parameter to ensure that they are physically equivalent? This is the central issue addressed here. The relation to the Seiberg-Witten map and the possible use to construct dualities are natural secondary issues that arise which will also be addressed, although not in complete generality here.

This paper is organized as follows. In section II we consider the general construction of a one parameter family of non-commutative, physically equivalent Hamiltonians. In section III and IV we apply this general construction to a particle in two dimensions moving in a constant magnetic field without interactions and in the presence of a harmonic potential, respectively. The construction is done to all orders in the non-commutative parameter. In section V the relation between this construction and the Seiberg-Witten map is discussed. In section VI we construct for an harmonic oscillator potential an approximate duality between the interacting commutative Hamiltonian and a non-interacting non-commutative Hamiltonian. Section VII contains our discussion and conclusions. Notational issues are summarized in appendix A.

II. GENERAL CONSIDERATIONS

We consider a non-relativistic particle moving in a plane under a potential V and coupled minimally to a $U(1)$ gauge field A. In commutative space the Hamiltonian reads ($\hbar = c = e = 1$)

$$H = \frac{(\mathbf{p}-\mathbf{A})^2}{2m} + V(x). \tag{1}$$

To go over to the non-commutative space we replace the commutative quantities by non-commutative ones, denoted by a hat, and introduce the star product, defined in the usual way

$$\left(\hat{f} \star \hat{g}\right)(x) = e^{\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha \partial'_\beta} \hat{f}(x)\hat{g}(x')\big|_{x'=x} \tag{2}$$

with θ the non-commutative parameter. We assume that there is no space time non-commutativity ($\theta^{0i} = 0$) and, for a planar system, the spatial part of the θ-matrix can be written as $\theta^{ij} = \theta\epsilon^{ij}$. The Schrödinger equation in non-commutative space therefore reads

$$\begin{aligned} i\frac{\partial\hat{\psi}(\mathbf{x},t)}{\partial t} &= \left[\frac{\left(\mathbf{p}-\hat{\mathbf{A}}\right)\star\left(\mathbf{p}-\hat{\mathbf{A}}\right)}{2\hat{m}} + \hat{V}(x)\right] \star \hat{\psi}(\mathbf{x},t) \\ &= \hat{H}\star\hat{\psi}(\mathbf{x},t) \equiv \hat{H}_{BS}(\theta)\hat{\psi}(\theta). \end{aligned} \tag{3}$$

Here $\hat{H}_{BS}(\theta)$ denotes the Hamiltonian after the star product has been replaced by a Bopp-shift, defined by [5, 7, 10]

$$\left(\hat{f}\star\hat{g}\right)(x) = \hat{f}\left(x - \frac{\theta}{2}\epsilon^{ij}p_j\right)\hat{g}(x). \tag{4}$$

Note that the quantities appearing in $\hat{H}_{BS}(\theta)$ are still the non-commutative ones.

The condition that the physics remains invariant under a change in θ requires that $\hat{H}_{BS}(\theta)$ and $\hat{H}_{BS}(0)$ are related by a unitary transformation,

$$\hat{H}_{BS}(\theta) = U(\theta)\hat{H}_{BS}(0)U^\dagger(\theta) \tag{5}$$

and that

$$\hat{\psi}(\theta) = U(\theta)\hat{\psi}(0)\ . \tag{6}$$

Differentiating (5) with respect to θ, we obtain

$$\frac{d\hat{H}_{BS}(\theta)}{d\theta} = [\eta(\theta), \hat{H}_{BS}(\theta)] \tag{7}$$

where

$$\eta(\theta) = \frac{dU(\theta)}{d\theta}U^\dagger(\theta) \tag{8}$$

is the generator of the unitary transformation relating the non-commutative Bopp-shifted Hamiltonian with the commutative Hamiltonian.

Let us consider under what conditions eq. (7) admits a solution for η. These conditions will, of course, provide us with the constraints on the parameterization of the non-commutative Hamiltonian necessary to ensure unitary equivalence, i.e., the existence of η. It is a simple matter to verify that eq. (7) admits a solution for η if and only if

$$\langle n,\theta|\frac{d\hat{H}_{BS}(\theta)}{d\theta}|n,\theta\rangle = 0 \quad , \quad \forall n \tag{9}$$

where $|n,\theta\rangle$ are eigenstates of $\hat{H}_{BS}(\theta)$, i.e.,

$$\hat{H}_{BS}(\theta)|n,\theta\rangle = E_n|n,\theta\rangle. \tag{10}$$

If eq. (9) holds, the off-diagonal part of η is uniquely determined by

$$\eta = \sum_{n\neq m} \frac{\langle n,\theta|\frac{d\hat{H}_{BS}}{d\theta}|m,\theta\rangle}{E_m - E_n}|n,\theta\rangle\langle m,\theta|, \tag{11}$$

while the diagonal part is arbitrary, reflecting the arbitrariness in the phase of the eigenstates. Here we have assumed no degeneracy in the spectrum of $\hat{H}_{BS}(\theta)$. The generalization to the case of degeneracies is straightforward.

The set of conditions (9) should be viewed as the set of conditions which determines the θ-dependency of the matrix elements of the non-commutative potential $\hat{V}$ and gauge field $\hat{A}$. Clearly one would expect that these matrix elements are under-determined, i.e., that not both

$\hat{V}$ and $\hat{A}$ are uniquely determined by them. Instead one can fix one of these and compute the other. For comparison with the Seiberg-Witten map it is therefore natural to take for $\hat{A}$ the non-commutative gauge field as determined from the Seiberg-Witten map. Note that this procedure implies that $\hat{V}$ will be gauge dependent.

Consider the Seiberg-Witten map for the non-commutative wave-function which reads to lowest order in θ [3] $\hat{\psi} = \psi - \frac{1}{2}\theta\epsilon^{ij}A_i\partial_j\psi$. Below we consider two dimensional systems in a constant magnetic field. Taking the symmetric gauge, the Seiberg-Witten map reduces to a θ dependent scaling transformation. Clearly this is not a unitary transformation and a unitary Seiberg-Witten map can be constructed as in [9]. However, a more convenient point of view, closer in spirit to the Seiberg-Witten map, would be to relax the condition of unitarity above. It therefore seems worthwhile, in particular to relate to the Seiberg-Witten map, to generalize the above considerations by relaxing the condition of unitarity.

This generalization is straightforward. We simply have to replace the unitary transformation in (5) and (6) by a general similarity transformation

$$\hat{H}_{BS}(\theta) = S(\theta)\hat{H}_{BS}(0)S^{-1}(\theta) \ , \tag{12}$$

while

$$\hat{\psi}(\theta) = S(\theta)\hat{\psi}(0), \tag{13}$$

and note that a new inner product $\langle\psi|\phi\rangle_T = \langle\psi|T|\phi\rangle$ can be defined such that $\hat{H}_{BS}(\theta)$ is hermitian with respect to it. In particular T is given by $T = (S^{-1})^\dagger S^{-1}$ and has the property $T\hat{H}_{BS}(\theta) = \hat{H}^\dagger_{BS}(\theta)T$. Under this prescription the same physics results. A detailed exposition of these issues can be found in [14].

Differentiating (12) with respect to θ, we obtain

$$\frac{d\hat{H}_{BS}(\theta)}{d\theta} = [\eta(\theta), \hat{H}_{BS}(\theta)] \tag{14}$$

where

$$\eta(\theta) = \frac{dS(\theta)}{d\theta}S^{-1}(\theta) \tag{15}$$

is now the generator of the similarity transformation relating the non-commutative Bopp-shifted Hamiltonian with the commutative Hamiltonian.

It is now a simple matter to verify that (9) gets replaced by

$$\langle n,\theta|T\frac{d\hat{H}_{BS}(\theta)}{d\theta}|n,\theta\rangle = 0 \quad , \quad \forall n \tag{16}$$

where $|n,\theta\rangle$ are eigenstates of $\hat{H}_{BS}(\theta)$ (note that the eigenvalues will be real as $\hat{H}_{BS}(0)$ is assumed hermitian and thus has real eigenvalues). As before, if eq. (16) holds the off-diagonal part of η is uniquely determined by

$$\eta = \sum_{n\neq m}\frac{\langle n,\theta|T\frac{d\hat{H}_{BS}}{d\theta}|m,\theta\rangle}{E_m - E_n}|n,\theta\rangle\langle m,\theta|T \tag{17}$$

while the diagonal part is arbitrary, reflecting the arbitrariness in the phase and now also the normalization of the eigenstates.

Under the above description, the Hamiltonians $\hat{H}_{BS}(\theta)$ and $\hat{H}_{BS}(0)$ are physically equivalent. There is, however, one situation in which this equivalence may break down and of which careful note should be taken. This happens when the similarity transformation $S(\theta)$ becomes singular for some value of θ, which will be reflected in the appearance of zero norm or unnormalizable states in the new inner product. Only values of θ which can be reached by integrating (15) from $\theta = 0$ without passing through a singularity, can be considered physically equivalent to the commutative system.

To solve (9) or (16) in general is of course impossible. Therefore we take a slightly different approach in what follows. We take an ansatz for η and solve (7) or (14) directly. The ansatz for η is again motivated by the Seiberg-Witten map. We have already noted above that in the cases of interest to us, i.e., two dimensional systems in constant magnetic fields, the Seiberg-Witten map for the non-commutative wave-function corresponds to a scaling transformation. Motivated by this we make the following ansatz

$$\eta(\theta) = f(\theta)r\partial_r = if(\theta)x.p \tag{18}$$

with f an arbitrary function to be determined. The finite form of this scaling transformation is simply obtained by integrating (15) to yield

$$S(\theta) = e^{i\left(\int_0^\theta f(\theta')d\theta'\right)x.p}. \tag{19}$$

Clearly this is not a unitary transformation and therefore falls in the class of more general transformations described above. Furthermore we note that the non-singularity of $S(\theta)$ requires that the integral $\int_0^\theta f(\theta')d\theta'$ exists.

III. FREE PARTICLE IN A CONSTANT MAGNETIC FIELD

In this section we apply the considerations discussed above to the case of a free particle ($\hat{V} = 0$) moving in a non-commutative plane in the presence of a constant non-commutative magnetic field. The Schrödinger equation is given by (3) with $\hat{V}$ set to zero.

In the symmetric gauge $\hat{A}_i = -\frac{\bar{B}(\theta)}{2}\epsilon_{ij}x_j$ [15] the Bopp-shifted Hamiltonian is easily found to be

$$\begin{aligned}\hat{H}_{BS}(\theta) &= \frac{\left(1+\frac{\bar{B}\theta}{4}\right)^2}{2\hat{m}(\theta)}\left(\mathbf{p} - \frac{1}{1+\frac{\bar{B}\theta}{4}}\mathbf{A}\right)^2 \\ &= \frac{1}{2M(\theta)}\left(p_x^2+p_y^2\right) + \frac{1}{2}M(\theta)\Omega(\theta)^2\left(x^2+y^2\right) \\ &\quad -\Omega(\theta)L_z \end{aligned} \tag{20}$$

where

$$\frac{1}{2M(\theta)} = \frac{\left(1+\frac{\bar{B}\theta}{4}\right)^2}{2\hat{m}(\theta)} \quad , \quad \frac{1}{2}M(\theta)\Omega(\theta)^2 = \frac{\bar{B}^2}{8\hat{m}(\theta)} \ . \tag{21}$$

Substituting the above form of the Hamiltonian in (7) with η as in (18), we obtain the following set of differential equations:

$$\frac{dM^{-1}(\theta)}{d\theta} = -2f(\theta)M^{-1}(\theta) \ , \tag{22}$$

$$\frac{d\left(M(\theta)\Omega(\theta)^2\right)}{d\theta} = 2M(\theta)\Omega(\theta)^2 f(\theta) \ , \tag{23}$$

$$\frac{d\Omega(\theta)}{d\theta} = 0. \tag{24}$$

Note that (24) ensures the stability of the energy spectrum, i.e the cyclotron frequency $\Omega(\theta) = \Omega(\theta = 0) = B/2m$, where $m = \hat{m}(\theta = 0)$. This is the physical input in our analysis and will play a very important role as we shall see later. Note that (24) follows trivially by combining the equations in (23). The above equations (21, 23, 24) immediately lead to

$$f(\theta) = \frac{1}{2M(\theta)}\frac{dM(\theta)}{d\theta} = \frac{\partial_\theta \bar{B}(\theta) - \frac{\bar{B}(\theta)^2}{4}}{2\bar{B}(\theta)\left(1+\frac{\theta\bar{B}(\theta)}{4}\right)} \tag{25}$$

which fixes f once $\bar{B}$ has been determined. As indicated before, we take $\hat{A}$ as the non-commutative gauge field determined from the Seiberg-Witten map. With this in mind we now proceed to determine $\bar{B}$. The Seiberg-Witten maps for $\hat{\psi}$ and $\hat{A}_k$ to lowest order in θ are [2, 3]

$$\begin{aligned} \hat{\psi} &= \psi - \frac{1}{2}\theta\epsilon^{ij}A_i\partial_j\psi \ , \\ \hat{A}_k &= A_k - \frac{1}{2}\theta\epsilon^{ij}A_i(\partial_j A_k + F_{jk}) \ . \end{aligned} \tag{26}$$

From the Seiberg-Witten transformation of the commutative gauge field A_k to the non-commutative one $\hat{A}_k$, one can easily see that a symmetric gauge configuration

$$A_i = -\frac{B}{2}\epsilon_{ij}x^j \tag{27}$$

with magnetic field $B = F_{12} = (\partial_1 A_2 - \partial_2 A_1)$, transforms to a symmetric gauge field configuration at the non-commutative level. Using the same notation as in (27) we write

$$\hat{A}_i = -\frac{\bar{B}}{2}\epsilon_{ij}x^j \tag{28}$$

where $\bar{B}$ is determined to leading order in θ from (26) to be

$$\bar{B} = B(1+\frac{3\theta B}{4}). \tag{29}$$

As pointed out in [15], $\bar{B}$ should not be identified with the non-commutative magnetic field $\hat{B}$, which has an additional Moyal bracket term $[\hat{A}_1, \hat{A}_2]_\star$:

$$\hat{B} = \hat{F}_{12} = \partial_1\hat{A}_2 - \partial_2\hat{A}_1 - i(\hat{A}_1\star\hat{A}_2 - \hat{A}_2\star\hat{A}_1) = \bar{B}(1+\frac{\theta\bar{B}}{4}). \tag{30}$$

This is precisely the same expression one gets if one applies the Seiberg-Witten map directly at the level of the field strength tensor, which is given by [2]:

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta\epsilon^{ij}F_{\mu i}F_{\nu j} \ . \tag{31}$$

Note that the expression (30) relating $\hat{B}$ with $\bar{B}$ is an exact one in contrast with (29) which relates $\bar{B}$ to B only up to leading order in θ. For a constant field configuration, the Seiberg-Witten equation for the field strength tensor can be integrated exactly to give the result [2]

$$\hat{B} = \frac{1}{1-\theta B}B. \tag{32}$$

From (30) and (32), we obtain a quadratic equation in $\bar{B}(\theta)$ that can be solved exactly to give

$$\bar{B}(\theta) = \frac{2}{\theta}\left[(1-\theta B)^{-1/2} - 1\right]. \tag{33}$$

The above expression for $\bar{B}(\theta)$ is exact up to all orders in θ. When substituted in (28) an expression, correct to all orders in θ, for the non-commutative gauge field $\hat{A}_i$ result

$$\hat{A}_i = -\frac{1}{\theta}\left[(1-\theta B)^{-1/2} - 1\right]\epsilon_{ij}x^j. \tag{34}$$

Substituting $\bar{B}$ from (33) into (25) yields

$$f(\theta) = \frac{\bar{B}(\theta)}{4} \ . \tag{35}$$

Upon differentiating (13) with respect to θ and using f from (35), we find that $\hat{\psi}(\theta)$ must satisfy the following equation:

$$\frac{d\hat{\psi}(\theta)}{d\theta} = \frac{\bar{B}(\theta)}{4} r \frac{d\hat{\psi}(\theta)}{dr} \ . \tag{36}$$

This result can now be compared to the corresponding Seiberg-Witten transformation rule for $\hat{\psi}$. The first of the Seiberg-Witten equations listed in (26) for an arbitrary $\theta + \delta\theta$ reads,

$$\hat{\psi}(\theta+\delta\theta) - \hat{\psi}(\theta) = -\frac{1}{2}\theta\epsilon^{ij}\hat{A}_i \star \partial_j\hat{\psi}(\theta). \tag{37}$$

Upon substituting $\hat{A}_i$ from (28), eq. (36) indeed results. Thus the transformation rule as obtained from the requirement of physical equivalence agrees with that of the Seiberg-Witten map.

Finally, substituting $\bar{B}(\theta)$ in the condition $\Omega = B/2m$ yields the following expression for $\hat{m}(\theta)$,

$$\hat{m}(\theta) = \frac{m}{1-\theta B} \ . \tag{38}$$

The above equation relates the non-commutative mass $\hat{m}(\theta)$ with the commutative mass m. This generalizes the result obtained in [6] to all orders in θ.

In a simple case such as this, one can of course solve the Schrödinger equation exactly. It is useful to see what the above procedure entails from this point of view. To solve for the eigenvalues and eigenfunctions of (20) is a standard procedure and for notational completeness we summarize the essential steps in appendix A. This results in the degenerate eigenvalue spectrum

$$\begin{gathered} E_{n_-,\ell} = 2\Omega\left(n_- + \frac{1}{2}\right) , \\ n_- = 0, 1 \ldots \quad ; \ \ell = -n_-, -n_- + 1, \ldots \ , \end{gathered} \tag{39}$$

where ℓ denotes the eigenvalues of the angular momentum operator L_3. The corresponding eigenstates are obtained by acting with the creation operators $b_\pm^\dagger$ defined in (A5) on the ground-state

$$\begin{aligned} \hat{\psi}(z,\bar{z};\theta) &= N \exp\left[-\frac{M\Omega}{2}\bar{z}z\right] \\ &= N \exp\left[-\frac{\bar{B}(\theta)}{4\left(1+\frac{\bar{B}(\theta)\theta}{4}\right)}\bar{z}z\right] \ . \end{aligned} \tag{40}$$

Comparing with our previous results, we note that eq. (24) ensures invariance of the spectrum under a change of θ. Furthermore direct inspection shows that the *unnormalised* ground-state and, subsequently, also all excited states satisfy the transformation rule (36). The fact that the unnormalised wave functions satisfy the transformation rule (36) is consistent with our earlier remarks on the non-unitary nature of the scaling transformation.

Finally, note that although the non-commutative parameters $\bar{B}(\theta)$ and $\hat{m}(\theta)$ have singularities at $\theta = 1/B$, these singularities cancel in the parameter Ω, which is by construction free of any singularities, i.e., the spectrum is not affected by this singularity. This is also reflected by the fact that the integral of f, as determined in (35), is free of this singularity. Thus, despite the appearance of this singularity in the parameters of the non-commutative Hamiltonian, there is no breakdown of the physical equivalence (see the discusion in section II).

IV. HARMONIC OSCILLATOR IN A CONSTANT MAGNETIC FIELD

In this section we include a harmonic oscillator potential $V = \lambda r^2$ in the commutative Hamiltonian (1). If the physical equivalence between the non-commutative and commutative Hamiltonians is indeed implementable through a scale transformation, we expect the potential to be form preserving (this is certainly not true for arbitrary potentials). We therefore extend to the non-commutative Hamiltonian by taking for the non-commutative potential in (3) $\hat{V} = \hat{\lambda}(\theta) r^2$, where the oscillator strength $\hat{\lambda}(\theta)$ has to be determined. Obviously we must also require that $\hat{\lambda}(\theta) = \lambda$ in the limit $\theta = 0$. Taking this form for the non-commutative Hamiltonian (3), the Bopp-shifted Hamiltonian is easily found to be

$$\begin{aligned} \hat{H}_{BS}(\theta) &= \frac{\left(1+\frac{\bar{B}\theta}{4}\right)^2}{2\hat{m}}\left(\mathbf{p} - \frac{1}{1+\frac{\bar{B}\theta}{4}}\mathbf{A}\right)^2 \\ &\quad + \hat{\lambda}(\theta)\left[\frac{\theta^2}{4}\left(p_x^2+p_y^2\right) + \left(x^2+y^2\right) - \theta L_z\right] \\ &= \frac{1}{2M}\left(p_x^2+p_y^2\right) + \frac{1}{2}M\Omega^2\left(x^2+y^2\right) \\ &\quad -\Lambda(\theta) L_z \end{aligned} \tag{41}$$

where

$$\begin{aligned} \frac{1}{2M} &= \frac{\left(1+\frac{\bar{B}\theta}{4}\right)^2}{2\hat{m}} + \frac{\hat{\lambda}\theta^2}{4} \ , \\ \frac{1}{2}M\Omega^2 &= \frac{\bar{B}(\theta)^2}{8\hat{m}(\theta)} + \hat{\lambda}(\theta) \ , \\ \Lambda(\theta) &= \left[\frac{M\Omega^2\theta}{2} + \frac{\bar{B}\left[1-\left(\frac{M\Omega\theta}{2}\right)^2\right]}{2\left(1+\frac{\bar{B}\theta}{2}\right)M}\right] \ . \end{aligned} \tag{42}$$

Here $\bar{B}(\theta)$ is again taken from the Seiberg-Witten map (33). Substituting the above form of the Hamiltonian in (7) with η as in (18), we obtain the following set of differential equations:

$$\frac{dM^{-1}(\theta)}{d\theta} = -2f(\theta)M^{-1}(\theta) \ , \tag{43}$$

$$\frac{d\left(M(\theta)\Omega(\theta)^2\right)}{d\theta} = 2M(\theta)\Omega(\theta)^2 f(\theta) \ , \tag{44}$$

$$\frac{d\Lambda(\theta)}{d\theta} = 0 \ . \tag{45}$$

Eq. (45) requires that $\Lambda(\theta)$ is independent of θ and hence we have the condition $\Lambda(\theta) = \Lambda(0) = B/2m$. Substituting the forms of $M(\theta)$ in terms of $\hat{m}(\theta)$ and $\hat{\lambda}(\theta)$, we obtain the following solution for $\hat{m}(\theta)$ in terms of $\hat{\lambda}(\theta)$:

$$\hat{m}(\theta) = \frac{m}{(1-\theta B)}\frac{B}{\left(B - 2m\theta\hat{\lambda}(\theta)\right)} \ . \tag{46}$$

The set of differential equations in (44) can also be combined to obtain

$$\frac{d\Omega^2}{d\theta} = 0 \ . \tag{47}$$

This shows that Ω is a constant and therefore we have

$$\Omega^2(\theta) = \Omega^2(\theta = 0) = \frac{B^2}{4m^2} + \frac{2\lambda}{m} \; . \tag{48}$$

Substituting $\Omega^2(\theta)$ in (42) and using (46), we get a quadratic equation for $\hat{\lambda}(\theta)$,

$$\left[B^3 + 8(1-\theta B)mB\hat{\lambda}(\theta) - 16(1-\theta B)m^2\theta\hat{\lambda}(\theta)^2\right] = B^3 + 8\lambda mB \; . \tag{49}$$

Solving for $\hat{\lambda}$ yields

$$\hat{\lambda}(\theta) = \frac{B}{4m\theta}\left[1 - \left(1 - \frac{8\lambda m\theta}{B(1-\theta B)}\right)^{\frac{1}{2}}\right] \; , \tag{50}$$

where we have taken the negative sign before the square root since with this choice we have $\hat{\lambda}(\theta = 0) = \lambda$.

With the value of $\bar{B}(\theta)$ fixed from the Seiberg-Witten map and $\hat{m}(\theta)$ and $\hat{\lambda}(\theta)$ determined as above, we can compute the value of $M(\theta)$ from (42) and subsequently the value of $f(\theta)$ from (44) as $f(\theta) = \frac{1}{2M(\theta)}\frac{dM(\theta)}{d\theta}$. This is a rather lenghthy expression which we do not need to list for our present purposes. What is important to note, however, is that once $f(\theta)$ is fixed, the transformation rule satisfied by $\hat{\psi}(\theta)$ is determined from (13) and that this transformation rule is not the same as the one derived from the Seiberg-Witten map (36). In fact, the transformation rule for $\hat{\lambda}(\theta)$ also turns out to be different from the Seiberg-Witten map. We discuss these points in more detail in the next section.

As a consistency check one can once again solve for the eigenvalues and eigenstates. The procedure is the same as in appendix A and one finds for the eigenvalues

$$E_{n_-,\ell} = 2\Omega\left(n_- + \frac{1}{2}\right) + (\Omega - \Lambda)\ell \; ,$$
$$n_- = 0, 1\ldots; \; \ell = -n_-, -n_- + 1, \ldots \; . \tag{51}$$

Note that the degeneracy in ℓ has been lifted. However, in the limit $\lambda = 0$, the energy spectrum given by (39) is recovered. The corresponding eigenstates are again obtained by acting with the creation operators $b^\dagger_\pm$ defined in (A5) on the ground-state

$$\hat{\psi}(z,\bar{z};\theta) = N\exp\left[-\frac{1}{4}\sqrt{\frac{2\bar{B}(\theta)^2 + 16\hat{\lambda}\hat{m}}{2\left(1 + \frac{\theta\bar{B}(\theta)}{4}\right)^2 + \theta^2\hat{\lambda}\hat{m}}}\,\bar{z}z\right] \; . \tag{52}$$

Once again we note that (44) and (45) ensures invariance of the spectrum under a change in θ. Using the values of $\bar{B}(\theta)$, $\hat{m}(\theta)$ and $\hat{\lambda}(\theta)$ as determined above, one finds that the *unnormalised* wave-functions indeed satisfy the transformation rule as determined by (13) and not the Seiberg-Witten transformation rule (36). Also, in the $\theta = 0$ limit, (52) smoothly goes over to the standard commutative result

$$\hat{\psi}(z,\bar{z},\theta = 0) = \psi(z,\bar{z}) = N\exp\left[-\frac{1}{4}\sqrt{B^2 + 8\lambda m}\,\bar{z}z\right] \; . \tag{53}$$

Finally we remark on the non-singularity of the scaling transformation $S(\theta)$. As already pointed out in section II this requires the existence of the integral of f, which in the present case is simply given by $\log(M(\theta)/m)/2$. This turns out to be free of singularities, although the non-commutative parameters again exhibit singularities at $\theta = 1/B$. As in the free case these singularities cancel in the parameters Ω and Λ which determine the physical spectrum.

V. RELATION TO SEIBERG-WITTEN MAP

In this section, we are going to discuss the relationship of the flow equations for $\hat{m}$ and $\hat{\lambda}$ obtained from the stability analysis of the previous section to the flow equation obtained from the Seiberg-Witten map. To that end, let us write down the $U(1)_\star$ gauge invariant action from which the $\star$ gauge covariant one-particle Schrödinger equation (3) follows as Euler-Lagrangian equation:

$$\hat{S} = \int d^3x\hat{\psi}^\dagger \star (i\hat{D}_0 + \frac{1}{2\hat{m}}\hat{D}_i \star \hat{D}_i + \hat{V}) \star \hat{\psi} \; . \tag{54}$$

The preservation of $U(1)_\star$ gauge invariance of the action requires that the potential $\hat{V}$ must transform adjointly under $\star$ gauge transformation

$$\hat{V}(x) \longrightarrow \hat{V}'(x) = \hat{U}(x)\star\hat{V}(x)\star\hat{U}^\dagger(x) \tag{55}$$

for $\hat{U}(x) \in U(1)_\star$. The reason for this is quite simple to see. If it were to remain invariant, this would have implied that the Moyal bracket between $\hat{V}$ and $\hat{U}$, $\forall$ $\hat{U} \in U(1)_\star$ vanishes ($[\hat{V},\hat{U}]_\star = 0$). Through Wigner-Weyl correspondence this in turn implies that $V_{\rm op}$ commutes with $U_{\rm op}$ at the operator level: $[V_{\rm op}, U_{\rm op}] = 0 \; \forall U_{\rm op}$. Applying Schur's lemma, assuming that $U_{\rm op}$ acts irreducibly, this indicates $V_{\rm op}$ =constant. Clearly this does not have the desired property. Now the Seiberg-Witten transformation property of $\hat{V}(x)$ can be easily obtained as

$$\hat{V}'(x) = \hat{V}(x) - \delta\theta\epsilon^{ij}\hat{A}_i \star \partial_j\hat{V}(x) \; , \tag{56}$$

which relates the non-commutative potential $\hat{V}(x;\theta) \equiv \hat{V}(x)$ for non-commutative parameter θ to the corresponding non-commutative potential $\hat{V}(x;\theta + \delta\theta) \equiv \hat{V}'(x)$ for non-commutative parameter $(\theta + \delta\theta)$. For the non-commutative gauge potential (28), this leads to the following differential equation

$$\frac{d\hat{V}(\theta)}{d\theta} = \frac{\bar{B}(\theta)}{2}r\frac{d\hat{V}(\theta)}{dr} \tag{57}$$

which can be solved by the method of seperation of variables [16], i.e. by taking $\hat{V}(r,\theta)=V(r)\hat{\lambda}_{sw}(\theta)$. We also have the boundary condition $\hat{\lambda}_{sw}(\theta=0)=\lambda$. Using this, (57) simplifies to

$$\frac{2}{\bar{B}(\theta)\hat{\lambda}_{sw}(\theta)}\frac{d\hat{\lambda}_{sw}(\theta)}{d\theta}=\frac{r}{V(r)}\frac{dV(r)}{dr}=k(=\text{constant}). \tag{58}$$

Solving we get

$$\begin{aligned} V(r) &= \lambda r^k\ , \\ \hat{\lambda}_{sw}(\theta) &= \lambda\exp\left[\frac{k}{2}\int_0^\theta d\theta^{'}\,\bar{B}(\theta^{'})\right] \\ &= \lambda\left(\frac{1+(1-\theta B)^{\frac{1}{2}}}{2}\right)^{-2k}\ . \end{aligned} \tag{59}$$

For $k=2$, we get the usual harmonic oscillator i.e.

$$\begin{aligned} \hat{V}(r,\theta) &= \hat{\lambda}_{sw}(\theta)r^2 \\ &= \lambda\left(\frac{1+(1-\theta B)^{\frac{1}{2}}}{2}\right)^{-4} r^2\ . \end{aligned} \tag{60}$$

If we now demand as in the free case that (52) satisfies (36) then the solution of (36) can also be found by taking the trial solution $\hat{\psi}(z,\bar{z};\theta)=N\exp\left(-\frac{\bar{z}z}{4}g(\theta)\right)$ subject to the boundary condition (53) at $\theta=0$. This leads to the solution

$$\hat{\psi}_{sw}(z,\bar{z};\theta)=N\exp\left[-\bar{z}z\frac{\sqrt{(B^2+8m\lambda)}}{\left((1-\theta B)^{\frac{1}{2}}+1\right)^2}\right]\ . \tag{61}$$

Comparing equations (52) and (61), we get an algebraic equation,

$$\frac{4\left(B^2+8m\lambda\right)^{\frac{1}{2}}}{\left[(1-\theta B)^{\frac{1}{2}}+1\right]^2}=\left[\frac{2\bar{B}^2(\theta)+16\hat{\lambda}_{sw}\hat{m}_{sw}}{2\left(1+\frac{\theta\bar{B}(\theta)}{4}\right)^2+\theta^2\hat{\lambda}_{sw}\hat{m}_{sw}}\right]^{\frac{1}{2}} \tag{62}$$

which leads to

$$\hat{\lambda}_{sw}\hat{m}_{sw}= \frac{4m\lambda\left[1+(1-\theta B)^{\frac{1}{2}}\right]^2}{(1-\theta B)\left(\{(1-\theta B)^{\frac{1}{2}}+1\}^4-\theta^2\left(B^2+8m\lambda\right)\right)}\ . \tag{63}$$

Substituting the value of $\hat{\lambda}_{sw}$ from (59) we obtain the value of $\hat{m}_{sw}$ as

$$\hat{m}_{sw}=\frac{m}{4\left(1-\theta B\right)}\frac{\left[1+(1-\theta B)^{\frac{1}{2}}\right]^6}{\left[\{(1-\theta B)^{\frac{1}{2}}+1\}^4-\theta^2\left(B^2+8m\lambda\right)\right]}\ . \tag{64}$$

The flow structure of $\hat{\lambda}_{sw}$ (59) and $\hat{m}_{sw}$ (64) in θ shows that the Seiberg-Witten flow is different (in the presence of interactions) from the flows obtained in the previous section (46) and (50) from the consideration of the stability of the spectrum, although the formal structure of the wave-functions $\hat{\psi}_{sw}$ (61) and $\hat{\psi}$ (52) are the same. Indeed, it can be checked easily and explicitly that the flow obtained here ((59) and (64)) from the Seiberg-Witten map is not spectrum preserving, as is the case with the flow of the previous section. This indicates that these flows are not equivalent or related in some simple way.

We have already seen that in absence of interaction ($\hat{\lambda}=0$) the non-commutative wave-function $\hat{\psi}_{sw}$ satisfies the Seiberg-Witten map, subject to the boundary condition (A7) at $\theta=0$, when $\hat{\psi}_{sw}$ becomes identifiable with the commutative wave-function ψ. Also, unlike its non-commutative counterpart $\hat{\psi}$, the commutative wave-function ψ does not have a flow of its own in θ. However, the situation changes drastically in the presence of interactions. To see this more clearly, let us consider the Schrödinger equation,

$$iD_0\psi=-\frac{1}{2m}D_iD_i\psi-\frac{i\theta}{2}\epsilon^{ij}F_{i0}D_j\psi \tag{65}$$

obtained from the $U(1)$ gauge invariant effective action in the presence of a background gauge field, derived in [6] to leading order in the non-commutative parameter θ. Note that the temporal component A_0 of the background gauge field can be regarded as $(-V)$, where V is the potential since this background gauge field is time independent. Indeed the Seiberg-Witten transformation property of both A_0 and V become identical, as can be seen from (56) and (26). This helps us to identify, again to leading order in θ, the corresponding Hamiltonian as

$$H=\frac{\left(\mathbf{p}-\mathbf{A}\right)^2}{2m}+V-\frac{\theta}{2}\epsilon^{ij}\partial_iV\left(p_j-A_j\right). \tag{66}$$

For a central potential $V(r)$, this simplifies in the symmetric gauge (27) to

$$H=\frac{\left(\mathbf{p}-\mathbf{A}\right)^2}{2m}+V-\frac{\theta}{2r}\frac{\partial V}{\partial r}\left(L_z-\frac{B}{2}r^2\right). \tag{67}$$

Again for a harmonic potential $V(r)=\lambda r^2$, this takes the form

$$H=\frac{\mathbf{p^2}}{2m}+\frac{B'^2}{8m}r^2-\tilde{\Lambda}L_z \tag{68}$$

where $B'=B\sqrt{1+\frac{8m\lambda}{B^2}(1+\frac{\theta B}{2})}$ and $\tilde{\Lambda}=\frac{B}{2m}+\theta\lambda$. Recognising that the structure of (68) is the same as that of (41), we can readily write down the ground state wave-function as

$$\begin{aligned} \psi_0(z,\bar{z};\theta) &= \exp\left(-\frac{B'(\theta)}{4}\bar{z}z\right) \\ &= \exp\left(-\frac{1}{4}\bar{z}z\sqrt{B^2+8m\lambda\left(1+\frac{\theta B}{2}\right)}\right)\ ; \\ & |\theta|<<1\ . \end{aligned} \tag{69}$$

This expression clearly reveals the fact that the commutative wave-function has a non-trivial flow in θ of its own, only in the presence of interaction ($\lambda \neq 0$) and the values of both non-commutative wave-functions $\hat{\psi}$, $\hat{\psi}_{sw}$ and the commutative one ψ coincide at $\theta = 0$. One can, in principle, determine the exact expression of this wave-function, valid upto all orders in θ, but we shall not require this here. In fact the wave-function (69) or higher angular momentum states $z^l\psi_0(z, \bar{z}; \theta)$ can be alternatively determined from perturbation theory applied to each angular momentum sector l for small θ and λ. However, one point that we would like to emphasise is that the Seiberg-Witten map does not map the non-commutative field $\hat{\psi}_{sw}(z, \bar{z}; \theta)$ at value θ to the corresponding one at the commutative level $\psi(z, \bar{z}; \theta)$; the Seiberg-Witten map or equivalently the Seiberg-Witten equation (36) only relates $\hat{\psi}(z, \bar{z}; \theta)$ to $\hat{\psi}(z, \bar{z}; \theta = 0) = \psi(z, \bar{z}; \theta = 0)$. Furthermore, the fact that the parameter $\hat{m}_{sw}(\theta)$ (64) does not reproduce the expression to leading order in θ, derived in [6] can be seen to follow from the observation that the parameter m was basically fixed by demanding the form invariance of the Schrödinger action which is equivalent to the stability analysis (in absence of interaction) we have carried out in the previous sections. Also observe that in [6] the "renormalised" mass parameter m does not get modified by the interaction term in any way, in contrast to both $\hat{m}_{sw}$ (64) and $\hat{m}$ (46). On the other hand, the commutative wave-function ψ in (69) gets modified in presence of interaction, as we mentioned above, in such a way that it has a non-trivial flow in θ. This is in contrast to the non-commutative wave-functions $\hat{\psi}$ (52) and $\hat{\psi}_{sw}$ (61) which have flows in θ even in absence of interactions. Finally, note that we have three versions of the Hamiltonians here with distinct transformations properties : (i) $\hat{H}$ occuring in (3) transforms adjointly under $U(1)_\star$ gauge transformation, (ii) H occuring in (66) transforms adjointly under ordinary $U(1)$ gauge transformation and (iii) the Bopp-shifted Hamiltonian $\hat{H}_{BS}$ occuring in (3) which, however, does not have any of these transformation properties under either type of gauge transformation as it was constructed just by disentangling the $\star$ product but retaining the non-commutative variables. In this context, it will be worthwhile to remind the reader that in order to have the symmetry under $\star$ gauge transformation we must have non-commutative variables composed through $\star$ product and to have the corresponding symmetry under ordinary gauge transformation, we must replace the non-commutative variables by commutative ones by making use of the Seiberg-Witten map apart from disentangling the $\star$ product as was done in [6]. Consequently, the issue of maintaining the gauge invariance/covariance is not relevant here, since we are dealing with $\hat{H}_{BS}$ in this paper.

VI. CONSTRUCTING DUALITIES

We have seen how physically equivalent families of non-commuting Hamiltonians can be constructed. In this construction θ simply plays the role of a parameter and subsequently, as the physics does not change, physical quantities can be computed with any value of this parameter. A natural question to pose, therefore, is whether there is any advantage in choosing a specific value of θ, i.e., is there any advantage in introducing non-commutativity in the first place. The motivation for asking this question was already outlined in section II, where it was pointed out that in some existing literature [12] the non-commutative quantum Hall system is considered a paradigm for the fractional quantum Hall effect which, however, requires the presence of interactions. If this interpretation is to be taken seriously a natural possibility that presents itself is that interacting commuting systems may in some approximation be equivalent to a particular non-interacting non-commutative system. If this turns out to be true, it would provide a new rational for the introduction of non-commutativity in quantum Hall systems. In this section we explore this possibility within a very simple setting.

We consider the non-commutative harmonic oscillator moving in a constant magnetic field discussed in section IV. After undoing the star product through a Bopp-shift we find the Hamiltonian

$$
\begin{aligned}
\hat{H}_{BS}(\theta) &= \frac{\mathbf{p}^2}{2M_0} + \frac{\mathbf{x}^2}{2} M_0 {\Omega_0}^2 - \Omega_0(\theta) L_z \\
&\quad + \hat{\lambda}\left(\frac{\theta^2}{4}\mathbf{p}^2 + \mathbf{x}^2 - \theta L_z\right) \\
&= \hat{H}_0 + \hat{V} \qquad (70)
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{1}{2M_0} &= \frac{\left(1 + \frac{\bar{B}\theta}{4}\right)^2}{2\hat{m}}, \\
\frac{1}{2} M_0 {\Omega_0}^2 &= \frac{\bar{B}(\theta)^2}{8\hat{m}(\theta)} \,. \qquad (71)
\end{aligned}
$$

To represent equivalent systems, the parameters $\bar{B}$, $\hat{m}$ and $\hat{\lambda}$ are parameterized as in (33), (46) and (50), respectively.

Naively one might argue that when the non-commutative coupling constant $\hat{\lambda}$ becomes small, the interaction term can be neglected on the non-commutative level. However, as this happens when θ becomes large ($\hat{\lambda} \sim 1/\theta$), one sees from the Bopp-shifted equivalent of the Hamiltonian that this is not true due to the θ dependency that is generated by the Bopp-shift. One therefore has to use a different criterion to decide when the interaction term $\hat{V}$ is small and can be neglected. One way is to introduce a norm on the space of operators and check that $\hat{V}$ is small in this norm. The trace norm $\mathrm{tr}(\hat{V}^\dagger \hat{V})$

is divergent and cannot be used; a regularization is required. An obvious alternative candidate to use is the following

$$Z(\theta) = \frac{\mathrm{tr}(\hat{V}^\dagger e^{-\beta\hat{H}_0}\hat{V})}{\mathrm{tr}e^{-\beta\hat{H}_0}} . \tag{72}$$

Here β plays the role of an energy cutt-off. It is clear that $Z(\theta)$ has all the properties of a norm, in particular $Z(\theta) = 0$ if and only if $\hat{V} = 0$. As remarked before, it is impossible to eliminate $\hat{V}$ completely, however, we can minimize $Z(\theta)$ with respect to θ and in doing this find the value of θ for which the non-commutative non-interacting Hamiltonian $\hat{H}_0$ gives the best approximation to the interacting Hamiltonian. Since the low-energy spectrum of $\hat{H}_0$ is biased in the norm (72), one can expect that the low-energy spectrum of $\hat{H}_0$ would give good agreement with the interacting spectrum, while the agreement will become worse as one moves up in the spectrum of $\hat{H}_0$. Before implementing this program, there is one further complication to take care of. Due to the degeneracy of $\hat{H}_0$ in the angular momentum, the norm (72) is still divergent when summing over angular momenta in the trace. However, since $\hat{V}$ is a central potential and subsequently different angular momentum sectors decouple, it is quite sufficient to implement the program above in each angular momentum sector seperately. Under minimization this will give rise to an angular momentum dependent value of θ, giving rise to a lifting in the degeneracy in angular momentum, which is what one would expect in the presence of interactions. To proceed we therefore replace (72) by

$$\begin{aligned} Z(\theta,\ell) &= \frac{\mathrm{tr}_\ell(\hat{V}^\dagger e^{-\beta\hat{H}_0}\hat{V})}{\mathrm{tr}_\ell e^{-\beta\hat{H}_0}} \\ &= \sum_{n_-=0}^{\infty} |\langle n_-,\ell|V|n_-,\ell\rangle|^2 e^{-\beta\Omega_0(2n_-+1)} \end{aligned} \tag{73}$$

where tr_ℓ denotes that the trace is taken over a fixed angular momentum sector, (39) was used and $|n_-,\ell\rangle$ denote the eigenstates of $\hat{H}_0$. This expression can be evaluated straightforwardly to yield

$$\begin{aligned} Z(\theta,\ell) &= \hat{\lambda}^2(\theta)\left[\Gamma(\theta)^2\left(1+\frac{2}{\sinh^2(\beta\Omega_0)}\right)\right. \\ &+ 2\ell\coth(\beta\Omega_0)\Gamma(\theta)\left(\Gamma(\theta)-\theta\right) \\ &+ \left.\ell^2\left(\Gamma(\theta)-\theta\right)^2\right] , \end{aligned} \tag{74}$$

where

$$\Gamma(\theta) = \frac{M_0\Omega_0\theta^2}{4} + \frac{1}{M_0\Omega_0} . \tag{75}$$

For $\beta >> 1/B$ one finds the value of θ that minimizes this expression to be

$$\theta(\ell) = \frac{2(1+\ell)}{B(1+2\ell)} \tag{76}$$

at which value $Z(\theta,\ell) \sim \frac{1}{B^2}$, which means that the potential at these values of θ can be treated as a correction of order $1/B$. The eigenvalues of $\hat{H}_0$ at these values of θ are easily evaluated to be

$$\begin{aligned} E_{n_-}(\ell) &= 2\Omega_0(\ell)(n_-+1/2) , \\ \Omega_0(\ell) &= \frac{B}{4m}\left(1+\sqrt{1+\frac{16\lambda m(\ell+1)}{B^2}}\right) . \end{aligned} \tag{77}$$

From the above considerations it is clear that the approximation is controlled by $1/B$. One therefore expects (77) to agree with the exact result (51), at least for the lowest eigenvalues, to order $1/B$. This indeed turns out to be the case. Expanding the lowest eigenvalues of (77) and (51) to leading order in $1/B$ one finds in both cases

$$E_0(\ell) = \frac{B}{2m} + \frac{2(\ell+1)\lambda}{B} . \tag{78}$$

This result suggests that it is indeed possible to trade the interactions for non-commutativity, at least in the lowest Landau level and for weak Landau level mixing (large B). It would, of course, be exceedingly naive to immediately extrapolate from the above to realistic quantum Hall systems. However, the above result does suggest a new paradigm for non-commutative quantum Hall systems worthwhile exploring. Within this paradigm interactions get traded, at least in the lowest Landau level, for non-commutativity, explaining the fractional filling fractions and emergence of composite fermions from a new perspective.

VII. DISCUSSION AND CONCLUSIONS

We have demonstrated how physically equivalent families of non-commutative Hamiltonians can be constructed. This program was explicitly implemented to all orders in the non-commutative parameter in the case of a free particle and harmonic oscillator moving in a constant magnetic field in two dimensions. It was found that this spectrum preserving map coincides with the Seiberg-Witten map in the case of no interactions, but not in the presence of interactions. A new possible paradigm for non-commutative quantum Hall systems was demonstrated in a simple setting. In this paradigm an interacting commutative system is traded for a weakly interacting non-commutative system, resulting in the same physics for the low energy sector. This provides a new rational for the introduction of non-commutativity in quantum Hall systems.

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APPENDIX A: EIGENVALUES AND EIGENSTATES OF THE FREE AND HARMONIC OSCILLATOR HAMILTONIANS

To solve for the eigenvalues and eigenstates of (20) one introduces creation and annihilation operators through the equations

$$\begin{aligned} b_x &= \sqrt{\frac{M\Omega}{2}}\left(x + \frac{ip_x}{M\Omega}\right), \quad b_x^\dagger = \sqrt{\frac{M\Omega}{2}}\left(x - \frac{ip_x}{M\Omega}\right), \\ b_y &= \sqrt{\frac{M\Omega}{2}}\left(y + \frac{ip_y}{M\Omega}\right), \quad b_y^\dagger = \sqrt{\frac{M\Omega}{2}}\left(y - \frac{ip_y}{M\Omega}\right). \end{aligned} \tag{A1}$$

In terms of these operators the Hamiltonian (20) takes the form,

$$H = \Omega\left(b_x^\dagger b_x + b_y^\dagger b_y + 1\right) - i\Omega\left(b_x b_y^\dagger - b_x^\dagger b_y\right) . \tag{A2}$$

From these creation and annihilation operators, written in a "cartesian basis", one can define corresponding creation and annihilation operators in a "circular basis" by making use of the transformations

$$\begin{aligned} b_+ &= \frac{1}{\sqrt{2}}\left(b_x - ib_y\right), \quad b_+^\dagger = \frac{1}{\sqrt{2}}\left(b_x^\dagger + ib_y^\dagger\right), \\ b_- &= \frac{1}{\sqrt{2}}\left(b_x + ib_y\right), \quad b_-^\dagger = \frac{1}{\sqrt{2}}\left(b_x^\dagger - ib_y^\dagger\right). \end{aligned} \tag{A3}$$

Using these, the Hamiltonian (A2) becomes

$$\begin{aligned} H &= \Omega\left(b_+^\dagger b_+ + b_-^\dagger b_- + 1\right) - \Omega\left(b_+^\dagger b_+ - b_-^\dagger b_-\right) \\ &= \Omega\left(n_+ + n_- + 1\right) - \Omega\left(n_+ - n_-\right) \\ &= 2\Omega\left(n_- + \frac{1}{2}\right) . \end{aligned} \tag{A4}$$

Note that the energy spectrum depends only on n_- and is independent of n_+. Therefore, it results in an infinite degeneracy in the energy spectrum. The above cancellation of the terms involving n_+ has taken place since the coefficients of n_+ are equal. This is also true in the limit $\theta = 0$. This feature does not persist in presence of interactions (see section 4).

Introducing complex coordinates $z = x + iy$ and $\bar{z} = x - iy$, (A3) takes the form

$$\begin{aligned} b_+ &= \frac{1}{2}\sqrt{M\Omega}\left[\bar{z} + \frac{2}{M\Omega}\partial_z\right], \quad b_+^\dagger = \frac{1}{2}\sqrt{M\Omega}\left[z - \frac{2}{M\Omega}\partial_{\bar{z}}\right], \\ b_- &= \frac{1}{2}\sqrt{M\Omega}\left[z + \frac{2}{M\Omega}\partial_{\bar{z}}\right], \quad b_-^\dagger = \frac{1}{2}\sqrt{M\Omega}\left[\bar{z} - \frac{2}{M\Omega}\partial_z\right]. \end{aligned} \tag{A5}$$

The ground state wave-function is annihilated by b_-, i.e. $b_-\hat{\psi}(z,\bar{z};\theta) = 0$. This immediately leads to the solution

$$\hat{\psi}_0(z,\bar{z};\theta) = N\exp\left[-\frac{M\Omega}{2}\bar{z}z\right] = N\exp\left[-\frac{\bar{B}}{4\left(1+\frac{\bar{B}\theta}{4}\right)}\bar{z}z\right].$$

Since $\bar{B}(\theta = 0) = B$, the above solution goes smoothly to the commutative result

$$\psi(z,\bar{z}) = N\exp\left[-\frac{B}{4}\bar{z}z\right] . \tag{A7}$$

This state is also annihilated by b_+ and therefore corresponds to zero angular momentum state, as the angular momentum operator $L_3 = (xp_y - yp_x)$ takes the following form in cartesian or circular basis

$$L_3 = i\left(b_x b_y^\dagger - b_x^\dagger b_y\right) = \left(b_+^\dagger b_+ - b_-^\dagger b_-\right). \tag{A8}$$

If this xy-plane is thought to be embedded in $3-d$ Euclidean space $\mathcal{R}^3$, then the other rotational generators L_1 and L_2 obtained by cyclic permutation would result in the standard angular momentum $SU(2)$ algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{A9}$$

One can, however, define the $SU(2)$ algebra using the creation and annihilation operators alone, which, in the cartesian basis, is given by

$$\begin{aligned} J_1 &= \frac{1}{2}\left(b_x^\dagger b_x - b_y^\dagger b_y\right), \\ J_2 &= \frac{1}{2}\left(b_x^\dagger b_y + b_y^\dagger b_x\right), \\ J_3 &= \frac{1}{2i}\left(b_x^\dagger b_y - b_y^\dagger b_x\right), \end{aligned} \tag{A10}$$

satisfying $[J_i, J_j] = i\epsilon_{ijk}J_k$. As one can easily verify, by computing the Poisson brackets of the generators with phase-space variables that J_1 generates rotation in (x, p_x) and (y, p_y) planes, J_y in (x, p_y) and (y, p_x) planes and J_z in (x, y) and (p_x, p_y) planes. Also note that L_3 is not identical to J_3 but differs by a factor of 2: $L_3 = 2J_3$.

The Casimir operator in terms of J_i representation now becomes

$$\vec{J}^2 = \frac{1}{4}\left(b_+^\dagger b_+ + b_-^\dagger b_-\right)\left(b_+^\dagger b_+ + b_-^\dagger b_- + 2\right). \tag{A11}$$

with eigenvalues $\vec{J}^2 = \frac{1}{4}(n_+ + n_-)(n_+ + n_- + 2)$. Defining $n_+ + n_- = 2j$, the Casimir becomes $\vec{J}^2 = j(j+1)$. Also, if the eigenvalues of J_3 is given by l', then the eigenvalues of L_3 will be given by $n_+ - n_- = 2l' = l\epsilon\mathcal{Z}$. Note that, like l', j also admits half-integral values. Finally, one can write down the eigenvalues (A4) as

$$E_{n_-} = \Omega\left(2j - 2l' + 1\right) = \Omega\left(2j - l + 1\right) \tag{A12}$$

which agrees with [7]. Any arbitrary state can now be obtained by repeated application of $b_\pm^\dagger$ on (A6) as

$$|n_-, l\rangle \sim \left(b_-^\dagger\right)^{n_-}\left(b_+^\dagger\right)^{l}\hat{\psi}_0(z,\bar{z};\theta) . \tag{A13}$$

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[15] We use $\bar{B}(\theta)$ to denote the non-commutative counterpart of B in (27). It should not be confused with the non-commutative magnetic field $\hat{B}$ as determined from the field strength (see (30)). In the limit $\theta = 0$, $\bar{B}(\theta) = B$.
[16] Such a seperation of variables can be made as one can expect that a commutative central potential goes over to another central potential at the non-commutative level.